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# A New Formulation of Relativistic Euler Flow: Miraculous Geo-Analytic Structures and Applications

#### Jared Speck

Vanderbilt University

October 27, 2020

Applications to Shock Waves

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# Main themes of the talk

#### • Solutions without symmetry

- We derived a new, geometric way of formulating relativistic Euler flow (joint with Disconzi)
- Key point: non-zero vorticity/entropy allowed
- Motivation: Christodoulou's work on irrotational shock formation and my previous non-relativistic work (with Luk in barotropic case)
- Potential applications: stable shock formation, low regularity, long-time behavior of solutions, dynamics with shocks, numerical simulations?

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Applications to Shock Waves

Looking Forward

# Relativistic Euler flow in Minkowski space

# ${\cal A}^lpha(ec{\Psi})\partial_lphaec{\Psi}={\sf 0}$

- $\vec{\Psi} = (h, u^0, u^1, u^2, u^3, s)$
- h = In H with H = enthalpy; u = four-velocity;
   s = entropy
- The system is quasilinear hyperbolic
- $\eta_{\alpha\beta}u^{\alpha}u^{\beta} = -1$ ,  $\eta = Minkowski metric$
- Equation of state p = p(ρ, s) closes the system (p = pressure, ρ = energy density)
- We assume c= sound speed  $:=\sqrt{rac{\partial p}{\partial a}}>0$  .
- Two propagation phenomena: sound waves and transporting of vorticity/entropy
- Neither the phenomena nor their coupling are visible
- s is crucial for the theory of solutions with shocks

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Looking Forward

## Geometric tensors associated to the flow

The four-velocity transports vorticity and entropy.

Definition (The four-velocity vectorfield)

# ${\it U}^lpha \partial_lpha$

The acoustical metric is tied to sound wave propagation.

Definition (The acoustical metric and its inverse)

$$\begin{split} \mathbf{g}_{\alpha\beta}(\vec{\Psi}) &:= c^{-2} \eta_{\alpha\beta} + (c^{-2} - 1) u_{\alpha} u_{\beta}, \\ (\mathbf{g}^{-1})^{\alpha\beta}(\vec{\Psi}) &= c^{2} (\eta^{-1})^{\alpha\beta} + (c^{2} - 1) u^{\alpha} u^{\beta} \end{split}$$

*u* is **g**-timelike and thus transverse to acoustically null hypersurfaces:

$$\mathbf{g}(u,u)=-1$$

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Applications to Shock Waves

Looking Forward

#### Covariant wave operator

#### Definition (Covariant wave operator)

For scalar-valued functions  $\phi$ , we define (as usual)

$$\Box_{\mathbf{g}}\phi := \frac{1}{\sqrt{|\mathsf{det}\mathbf{g}|}}\partial_{\alpha}\left\{\sqrt{|\mathsf{det}\mathbf{g}|}(\mathbf{g}^{-1})^{\alpha\beta}\partial_{\beta}\phi\right\}$$

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#### Additional fluid variables

Definition (The *u*-orthogonal vorticity of a one-form)

$$\mathsf{vort}^{lpha}(\mathit{V}) := -\epsilon^{lphaeta\gamma\delta}\mathit{u}_{eta}\partial_{\gamma}\mathit{V}_{\delta}$$

Definition (Vorticity vectorfield)

$$\varpi^{\alpha} := \operatorname{vort}^{\alpha}(Hu) = -\epsilon^{\alpha\beta\gamma\delta} u_{\beta} \partial_{\gamma}(Hu_{\delta})$$

Definition (Entropy gradient one-form)

$$S_{lpha}:=\partial_{lpha} {f s}$$

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Looking Forward

# Modified fluid variables

- Exhibit improved regularity
- Solve PDEs with good quasilinear null structure with respect to g

#### Definition (Modified fluid variables)

 $\begin{aligned} \mathcal{C}^{\alpha} &:= \operatorname{vort}^{\alpha}(\varpi) + c^{-2} \epsilon^{\alpha\beta\gamma\delta} u_{\beta}(\partial_{\gamma}h) \varpi_{\delta} \\ &+ (\theta - \theta_{jh}) S^{\alpha}(\partial_{\kappa}u^{\kappa}) + (\theta - \theta_{jh}) u^{\alpha}(S^{\kappa}\partial_{\kappa}h) \\ &+ (\theta_{jh} - \theta) S^{\kappa}((\eta^{-1})^{\alpha\lambda}\partial_{\lambda}u_{\kappa}), \\ \mathcal{D} &:= \frac{1}{n} (\partial_{\kappa}S^{\kappa}) + \frac{1}{n} (S^{\kappa}\partial_{\kappa}h) - \frac{1}{n} c^{-2} (S^{\kappa}\partial_{\kappa}h) \end{aligned}$ 

 Temperature θ(h, s) and number density n(h, s) determined by equation of state

• 
$$\theta_{;h} := \frac{\partial}{\partial h} \theta$$

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# Null forms relative to g

#### Definition (Null forms relative to g)

$$\begin{split} \mathbb{Q}^{(\mathbf{g})}(\partial\phi,\partial\widetilde{\phi}) &:= (\mathbf{g}^{-1})^{\alpha\beta}\partial_{\alpha}\phi\partial_{\beta}\widetilde{\phi},\\ \mathbb{Q}_{(\alpha\beta)}(\partial\phi,\partial\widetilde{\phi}) &:= \partial_{\alpha}\phi\partial_{\beta}\widetilde{\phi} - \partial_{\alpha}\widetilde{\phi}\partial_{\beta}\phi \end{split}$$

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#### Purpose of new formulation

# The new formulation allows for the application of geometric techniques from mathematical GR and nonlinear wave equations.

- Big new issue compared to waves:
  - The interaction of wave and transport phenomena, especially from the perspective of regularity and decay.
    - "multiple characteristic speeds"

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## A new formulation of relativistic Euler

Theorem (JS with M. Disconzi)

For  $\Psi \in \vec{\Psi} := (h, u^0, u^1, u^2, u^3, s)$ ,  $\Omega :=$  combinations of null forms, regular solutions satisfy, up to lower-order terms:

$$\begin{split} & \Box_{\mathbf{g}(\vec{\psi})} \Psi = \mathcal{C} + \mathcal{D} + \mathfrak{Q}(\partial \vec{\Psi}, \partial \vec{\Psi}), \\ & u^{\kappa} \partial_{\kappa} \varpi^{\alpha} = \partial \vec{\Psi}, \\ & u^{\kappa} \partial_{\kappa} S^{\alpha} = \partial \vec{\Psi} \end{split}$$

 Formally, C, D ~ ∂∂Ψ, but they are actually better from various points of view. In fact, ∂π, ∂S are better:

 $egin{aligned} &\partial_{lpha}arpi^{lpha} = arpi \cdot \partialec{\psi}, \ &\mathcal{O}^{lpha} \partial_{lpha} \mathcal{C}^{lpha} = \Omega(\partial arpi, \partialec{\psi}) + \Omega(\partial S, \partialec{\psi}) \ &+ \partialec{\psi} \cdot \mathcal{C} + \partialec{\psi} \cdot \mathcal{D} + \Omega(\partialec{\psi}, \partialec{\psi}) \end{aligned}$ 

 $u^{\kappa}\partial_{\kappa}\mathcal{D} = \Omega(\partial S, \partial ec{\Psi}) + \Omega(\partial ec{\Psi}, \partial ec{\Psi}),$ vort<sup>e</sup>(S) = 0

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• Formally,  $C, D \sim \partial \partial \vec{\Psi}$ , but they are actually better from various points of view. In fact,  $\partial \varpi, \partial S$  are better:

$$\begin{split} \partial_{\alpha} \varpi^{\alpha} &= \varpi \cdot \partial \vec{\Psi}, \\ u^{\kappa} \partial_{\kappa} \mathcal{C}^{\alpha} &= \mathfrak{Q}(\partial \varpi, \partial \vec{\Psi}) + \mathfrak{Q}(\partial S, \partial \vec{\Psi}) \\ &+ \partial \vec{\Psi} \cdot \mathcal{C} + \partial \vec{\Psi} \cdot \mathcal{D} + \mathfrak{Q}(\partial \vec{\Psi}, \partial \vec{\Psi}) \end{split}$$

$$\begin{split} u^{\kappa}\partial_{\kappa}\mathcal{D} &= \mathfrak{Q}(\partial \boldsymbol{S}, \partial \vec{\Psi}) + \mathfrak{Q}(\partial \vec{\Psi}, \partial \vec{\Psi}),\\ \text{vort}^{\alpha}(\boldsymbol{S}) &= \boldsymbol{0} \end{split}$$

Looking Forward

# L<sup>2</sup> regularity via div-curl-transport

- In non-relativistic flow, the div-curl part is along  $\Sigma_t$ .
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   u<sub>α</sub>∂ω<sup>α</sup> = −(∂u<sub>α</sub>)ω<sup>α</sup>).
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- Stable shock formation without symmetry (à la Christodoulou and my work with Luk in the non-relativistic case). Null structure is crucial.
- Thesis work in progress by Sifan Wu: low regularity sound waves (à la my work with Disconzi, Luo, Mazzone and Wang's work in the non-relativistic case). Null structure not needed.
- Small-time extension of the solution past the first shock (Christodoulou solved the Shock Development Problem in the irrotational case). Null structure is crucial.
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The new formulation opens the door for several key applications with vorticity and entropy, some of which have been achieved in the non-relativistic case:

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Looking Forward

# Nonlinear geometric optics

- Potential applications would require nonlinear geometric optics.
- New formulation allows for sharp implementation of nonlinear geometric optics.
- Implemented via an acoustic eikonal function U:

 $(\mathbf{g}^{-1})^{lphaeta}(ec{\psi})\partial_lpha U\partial_eta U=0, \qquad \quad \partial_t U>0$ 

- Level sets  $C_U$  of U are g-null hypersurfaces.
- Play a critical role in many delicate local and global results for wave equations.
- The regularity theory of U is difficult, tensorial, influenced by the Euler solution, especially the vorticity and entropy.

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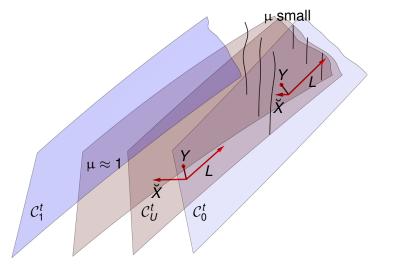
Intro New Formulation

Nonlinear Geometric Optics

Applications to Shock Waves

Looking Forward

# g-null hypersurfaces close to plane symmetry

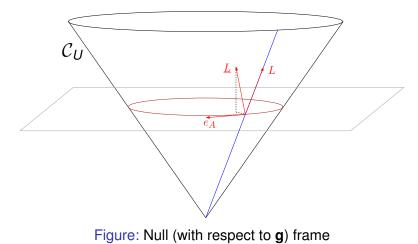


Applications to Shock Waves

Looking Forward

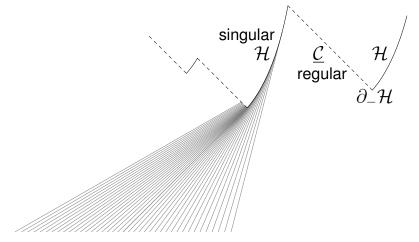
### Acoustic null frame

An acoustic null frame  $\{L, \underline{L}, e_1, e_2\}$ :



Looking Forward

# Christodoulou's sharp picture of relativistic Euler shock formation (irrotational case)



#### Figure: The maximal development

Intro	New Formulation

Applications to Shock Waves

Looking Forward

#### Model problem

$$\mathbf{g}(\Psi) = -dt \otimes dt + (1 + \Psi)^{-2} \sum_{a=1}^{3} dx^{a} \otimes dx^{a}$$

$$\Box_{\boldsymbol{g}(\boldsymbol{\Psi})}\boldsymbol{\Psi}=\boldsymbol{0}$$

 $(t, x^1)$  plane symmetry, define null vectorfields  $L := \partial_t + (1 + \Psi)\partial_1, \qquad \underline{L} := \partial_t - (1 + \Psi)\partial_1.$ 

The wave equation can be expressed as:



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$$egin{aligned} & L(\underline{L}\Psi) = \underbrace{rac{1}{2(1+\Psi)}(\underline{L}\Psi)^2}_{ ext{causes Riccati-type blowup}} + rac{5}{2(1+\Psi)}(\underline{L}\Psi)L\Psi, \ & L(L\Psi) = -rac{1}{2(1+\Psi)}(L\Psi)^2 + rac{5}{2(1+\Psi)}(\underline{L}\Psi)L\Psi. \end{aligned}$$

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Looking Forward

#### Eikonal functions regularize the problem

Define eikonal functions  $U, \underline{U}$  by:

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 $U(0, x^1) = -x^1,$   $\underline{U}(0, x^1) = x^1.$ 

Then in  $(U, \underline{U})$  coordinates, the wave equation becomes

$$\frac{\partial}{\partial \underline{U}} \frac{\partial}{\partial U} \Psi = \frac{2}{(1+\Psi)} \frac{\partial}{\partial \underline{U}} \Psi \cdot \frac{\partial}{\partial U} \Psi$$

 $\implies$  For "many" data, solution remains smooth in  $(U, \underline{U})$  coordinates!

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#### Eikonal functions regularize the problem

Define eikonal functions  $U, \underline{U}$  by:

$$LU = 0,$$
  $\underline{LU} = 0,$   
 $U(0, x^1) = -x^1,$   $\underline{U}(0, x^1) = x^1.$ 

#### Then in $(U, \underline{U})$ coordinates, the wave equation becomes

$$\boxed{\frac{\partial}{\partial \underline{U}} \frac{\partial}{\partial U} \Psi = \frac{2}{(1+\Psi)} \frac{\partial}{\partial \underline{U}} \Psi \cdot \frac{\partial}{\partial U} \Psi}$$

 $\implies$  For "many" data, solution remains smooth in  $(U, \underline{U})$  coordinates!

Looking Forward

#### Singularity is visible in standard coordinates

Set 
$$\mu := \frac{1}{\underline{L}U}$$
 so that  $\mu \underline{L} = \frac{\partial}{\partial U}$ . Set  $\underline{\mu} := \frac{1}{\underline{L}\underline{U}}$  so that  $\underline{\mu}\underline{L} = \frac{\partial}{\partial \underline{U}}$ .

Evolution equations for  $\mu, \mu$ :



•  $L\Psi = \frac{1}{\mu} \frac{\partial}{\partial U} \Psi \implies |L\Psi| \rightarrow \infty$  when  $\mu \downarrow 0$ •  $L\Psi = \frac{1}{\mu} \frac{\partial}{\partial U} \Psi \implies |L\Psi|$  remains bounded if  $\mu > 0$ •  $\mu \rightarrow \langle \partial \mu \rangle \langle \partial \mu \rangle$ 

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$$\begin{aligned}
\frac{\partial}{\partial \underline{U}} \mu &= \underbrace{-\frac{\underline{\mu}}{(1+\Psi)}}_{\text{Can drive } \mu \downarrow 0} \frac{\partial}{\partial U} \Psi - \underbrace{-\frac{\mu}{(1+\Psi)}}_{\partial \underline{U}} \frac{\partial}{\partial \underline{U}} \Psi, \\
\frac{\partial}{\partial U} \underline{\mu} &= -\frac{\mu}{(1+\Psi)}}_{\partial \underline{U}} \Psi - \underbrace{-\frac{\mu}{(1+\Psi)}}_{\partial \underline{U}} \frac{\partial}{\partial U} \Psi
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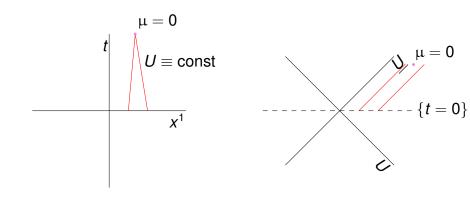
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# Regularizing the singularity



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Looking Forward

#### Significance of null forms

For null forms  $\Omega^{Null}$ , in plane symmetry,  $\Box_{g(\Psi)}\Psi = \Omega^{Null}(\partial \Psi, \partial \Psi)$  can be written as

$$\frac{\partial}{\partial \underline{U}} \frac{\partial}{\partial U} \Psi = f(\Psi) \frac{\partial}{\partial \underline{U}} \Psi \cdot \frac{\partial}{\partial U} \Psi.$$

This equation can be treated as before.

In contrast, for a typical quadratic term  $\Omega^{Bad}(\partial \Psi, \partial \Psi) = \partial \Psi \cdot \partial \Psi, \square_{g(\Psi)} \Psi = \Omega^{Bad}(\partial \Psi, \partial \Psi)$  can be written as

$$\frac{\partial}{\partial \underline{U}} \frac{\partial}{\partial U} \Psi = \frac{\mu}{\mu} f(\Psi) \frac{\partial}{\partial U} \Psi \cdot \frac{\partial}{\partial U} \Psi + \cdots$$

#### The bad factor of $\frac{1}{n}$ spoils the previous analysis as $\mu_{\rm s}$ is $\mu_{\rm s}$ , $\mu_{\rm s}$

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Looking Forward

#### Basic proof strategy in 1 + 3 dimensions

- Supplement *t* and *U* with geometric angular coordinates θ ∈ S<sup>2</sup>
- Prove that the solution remains smooth relative to (t, U, θ) coordinates
- Recover the blowup as a degeneracy between (t, U, ϑ) and rectangular coordinates

The degeneracy is signified by the vanishing of the inverse foliation density:

$$\mu = -rac{1}{(g^{-1})^{lphaeta}\partial_lpha t\partial_eta U} > 0$$

Looking Forward

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- All known well-posedness results rely on L<sup>2</sup>-based Sobolev spaces;
- Energy estimates are very difficult in regions where  $\mu\downarrow 0$
- High-order geometric energies can blow up: <sup>High</sup>(t) ≲ (min<sub>Σ</sub>, μ)<sup>-P</sup>, P ≈ 10
- The possible high-order energy blowup makes it difficult to show that the solution's mid-order geometric derivatives are bounded
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Looking Forward ●○

# Directions to consider

- Does Einstein–Euler exhibit similar good structures?
- Shock formation for Einstein–Euler
- Same questions for MHD, viscous relativistic Euler
- Same questions for more complicated multiple speed systems: elasticity, crystal optics, nonlinear electromagnetism,..., which take the form:

$$h^{lphaeta}_{AB}(\partial\Phi)\partial_lpha\partial_eta\Phi^B=0$$

- Solve past the shock, locally (shock development problem à la Christodoulou)
- Long-time behavior of solutions with shocks (at least in a perturbative regime)
- Long-time behavior of vorticity
- Useful for numerical simulations?

Applications to Shock Waves

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- Same questions for more complicated multiple speed systems: elasticity, crystal optics, nonlinear electromagnetism,..., which take the form: h<sup>αβ</sup><sub>AP</sub>(∂Φ)∂<sub>α</sub>∂<sub>B</sub>Φ<sup>B</sup> = 0

- Solve past the shock, locally (shock development problem à la Christodoulou)
- Long-time behavior of solutions with shocks (at least in a perturbative regime)
- Long-time behavior of vorticity
- Useful for numerical simulations?

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Looking Forward ●○

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Nonlinear Geometric Optics

Applications to Shock Waves

Looking Forward

# Thank you